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Analysis of a new model of H1N1 spread: Model obtained via Mittag-Leffler function

Badr Saad T Alkahtani[1], Ilknur Koca[2] and Abdon Atangana[3]

Abstract

In the recent decades, many physical problems were modelled using the concept of power law within the scope of fractional differentiations. When checking the literature, one will see that there exist many formulas of power law, which were built for specific problems. However, the main kernel used in the concept of fractional differentiation is based on the power law function $x^{-\lambda}$ It is quick important to note all physical problems, for instance, in epidemiology. Therefore, a more general concept of differentiation that takes into account the more generalized power law is proposed. In this article, the concept of derivative based on the Mittag-Leffler function is used to model the H1N1. Some analyses are done including the stability using the fixed-point theorem.

Keywords

H1N1 model, Mittag-Leffler function, fixed-point theorem, iterative method

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Introduction

In the last decade, it was found that many physical problems' behaviour follows the power law. Also, some powerful methods and mathematical models are shown in the fractional order concept from all over the world.[1–5] However, for a specific physical problem, there is a corresponding power law that can be used to describe the future behaviour of the observed fact.[6–10] This application of power law is found in many branches of science and technology. For instance, in statistics, a power law is used as a functional correlation connecting two quantities, anywhere a relative change in one quantity results in a comparative relative alter in other quantity, independent of the original size of those quantities, one quantity varies as power of another. Nonetheless, the concept of fractional calculus is based on the concept of power law, but the power law used within this field is nothing more than $x^{-\lambda}$. The concept of fractional differentiation has been used in almost all the field of science, engineering, technology and others. However, all these problems for which this concept was applied do not necessarily follow the power law based on the function $x^{-\lambda}$. Here is the failure of the power law function $x^{-\lambda}$ in statistics, a power law function $x^{-\lambda}$ has a well-defined average over a range of $[1, \infty]$ only if $\alpha > 2$ and it has finite variance only when $\alpha > 3$, but most applications of fractional differentiation are done only when $0 < \alpha < 1$.[6–10] To further broaden the scope of fractional calculus, Caputo and Fabrizio

[1]Department of Mathematics, College of Sciences, King Saud University, Riyadh, Saudi Arabia
[2]Department of Mathematics, Faculty of Sciences, Mehmet Akif Ersoy University, Burdur, Turkey
[3]Institute for Groundwater Studies, Faculty of Natural and Agricultural Sciences, University of the Free State, Bloemfontein, South Africa

Corresponding author:
Abdon Atangana, Institute for Groundwater Studies, Faculty of Natural and Agricultural Sciences, University of the Free State, Bloemfontein 9300, South Africa.
Email: abdonatangana@yahoo.fr

suggested a fractional derivative based on the exponential decay law which is a generalized power law function. However, many other researchers testified that the Caputo–Fabrizio operator is nothing more than a filter with a fractional regulator. They based their argument upon the fact that the kernel used in this design is not non-local; in addition, the integral associate is the average of the given function and its integral. In many solutions of the fractional differentiation based on the power law $x^{-\lambda}$, the Mittag-Leffler function is mostly present. The Mittag-Leffler function is of course the more generalized exponential function; in addition, it is also a non-local kernel.[11–14] To solve the failures of the Caputo–Fabrizio derivative, the fractional derivative based on the Mittag-Leffler function was introduced and used in some new problem with great success.[15–19] It is important to know that where the power based on $x^{-\lambda}$ function relax then raise the Mittag-Leffler function more complex problems. In this article, we apply the newly established derivative with non-singular and non-local kernel by Atangana and Baleanu to model the spread of influenza.

New fractional differentiation based on Mittag-Leffler function

We present in this section the novel fractional operators based on the Mittag-Leffler function. The novel fractional derivatives are known as Atangana–Baleanu fractional derivative in Caputo sense (ABC) and Atangana–Baleanu fractional derivative in Riemann–Liouville sense (ABR). These definitions can be found in the study of Atangana and colleagues;[11,12] we shall therefore present the definition as it is in the initial work.[11,12]

Definition 1. Let $f \in H^1(a,b),\ b>a$ and $\alpha \in [0,1]$, then the definition of the new fractional derivative (ABC) is given as

$$ {}^{ABC}_{\ a}D_t^{\alpha}(f(t)) = \frac{B(\alpha)}{1-\alpha}\int_a^t f'(x)E_\alpha\left[-\alpha\frac{(t-x)^\alpha}{1-\alpha}\right]dx \quad (1)$$

In their work, they clarified that the function B has the same properties as that of Caputo and Fabrizio's definition.

Definition 2. Let $f \in H^1(a,b),\ b>a$ and $\alpha \in [0,1]$, and not necessarily differentiable, then the definition of the new fractional derivative (ABR) is given as

$$ {}^{ABR}_{\ a}D_t^{\alpha}(f(t)) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_a^t f(x)E_\alpha\left[-\alpha\frac{(t-x)^\alpha}{1-\alpha}\right]dx \quad (2)$$

Definition 3. The fractional integral associate to the new fractional derivative with non-local kernel is defined as

$$ {}^{AB}_{\ a}I_t^{\alpha}\{f(t)\} = \frac{1-\alpha}{B(\alpha)}f(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_a^t f(y)(t-y)^{\alpha-1}dy \quad (3)$$

Here also they reported that when alpha tends to zero, the initial function is obtained, and when alpha tends to 1, the classical integral is obtained.[11,12]

Analysis of existence and unicity of the new system

Let us redefine the classical model of N1H1 spread by replacing the time derivative by time fractional derivative, and we shall recall that the reason for the modification has been presented in the 'Introduction' section. Nevertheless, it is important noting that the concept of local derivative that is used to describe the rate of change has failed to model accurately some complex real-world problems. Due to this failure, the concept of fractional differentiation based on the convolution of $x^{-\alpha}$ was introduced, and also failed in some cases due to the disc of convergence of this function. The Mittag-Leffler function, that is the more generalized version, can therefore be used in order to handle more physical problems

$$\begin{aligned}
{}^{ABC}_{\ 0}D_t^{\alpha}S(t) &= -\beta S(t)\frac{qE(t)+I(t)+A(t)}{N(t)}\\
{}^{ABC}_{\ 0}D_t^{\alpha}E(t) &= \beta S(t)\frac{qE(t)+I(t)+A(t)}{N(t)} - \delta E(t)\\
{}^{ABC}_{\ 0}D_t^{\alpha}I(t) &= p\delta E(t) - \gamma_1 I(t)\\
{}^{ABC}_{\ 0}D_t^{\alpha}A(t) &= (1-p)\delta E(t) - \gamma_2 A(t)\\
{}^{ABC}_{\ 0}D_t^{\alpha}R(t) &= \gamma_1 I(t) + \gamma_2 A(t)\\
{}^{ABC}_{\ 0}D_t^{\alpha}C(t) &= p\delta E(t)
\end{aligned} \quad (4)$$

A very important fact in differential calculus is to prove the existence and the uniqueness of the solution of a given problem; therefore, in this section, we aim to prove the existence of solutions for the new model. The system state is made up of S, E, I, A, R, C. The constants used in this model are the same like in Tan et al.[20] The above system is equivalent to Volterra type, where the integral is that of Atangana–Baleanu fractional integral. We shall recall that the Atangana–Baleanu fractional integral of a function $f(t)$ is the average of the function $f(t)$ and the Riemann–Liouville fractional integral. The proof is shown in theorem 1.

Theorem 1. The following time fractional ordinary differential equation

$$ {}_{0}^{ABC}D_{t}^{\alpha}(f(t)) = u(t) - u(0) \tag{5}$$

has a unique solution which takes the inverse Laplace transform and uses the convolution theorem below[12]

$$ f(t) - f(0) = \frac{1-\alpha}{B(\alpha)} u(t) + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_{a}^{t} u(y)(t-y)^{\alpha-1} dy \tag{6}$$

With the theorem above, the system is equivalent to the following

$$ \begin{cases} S(t) - g_1(t) = \frac{1-\alpha}{B(\alpha)} \left\{ -\beta S(t) \frac{qE(t)+I(t)+A(t)}{N(t)} \right\} \\ \quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-y)^{\alpha-1} \left\{ -\beta S(y) \frac{qE(y)+I(y)+A(y)}{N(y)} \right\} dy \\ E(t) - g_2(t) = \frac{1-\alpha}{B(\alpha)} \left\{ \beta S(t) \frac{qE(t)+I(t)+A(t)}{N(t)} - \delta E(t) \right\} \\ \quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-y)^{\alpha-1} . \left\{ -\beta S(y) \frac{qE(y)+I(y)+A(y)}{N(y)} - \delta E(y) \right\} dy \\ I(t) - g_3(t) = \frac{1-\alpha}{B(\alpha)} \{ p\delta E(t) - \gamma_1 I(t) \} \\ \quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-y)^{\alpha-1} \{ p\delta E(y) - \gamma_1 I(y) \} dy \\ A(t) - g_4(t) = \frac{1-\alpha}{B(\alpha)} \{ (1-p)\delta E(t) - \gamma_2 A(t) \} \\ \quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-y)^{\alpha-1} \{ (1-p)\delta E(y) - \gamma_2 A(y) \} dy \\ R(t) - g_5(t) = \frac{1-\alpha}{B(\alpha)} \{ \gamma_1 I(t) + \gamma_2 A(t) \} \\ \quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-y)^{\alpha-1} \{ \gamma_1 I(y) + \gamma_2 A(y) \} dy \\ C(t) - g_6(t) = \frac{1-\alpha}{B(\alpha)} \{ p\delta E(t) \} + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-y)^{\alpha-1} \{ p\delta E(y) \} dy \end{cases} \tag{7}$$

A possibility of converting the above system to iterative routine is given below

$$ \begin{cases} S_0(t) = g_1(t) \\ E_0(t) = g_2(t) \\ I_0(t) = g_3(t) \\ A_0(t) = g_4(t) \\ R_0(t) = g_5(t) \\ C_0(t) = g_6(t) \end{cases} \tag{8}$$

$$ S_{n+1}(t) = \frac{1-\alpha}{B(\alpha)} \left\{ -\beta S_n(t) \frac{qE_n(t)+I_n(t)+A_n(t)}{N_n(t)} \right\} + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-y)^{\alpha-1} \left\{ -\beta S_n(y) \frac{qE_n(y)+I_n(y)+A_n(y)}{N_n(y)} \right\} dy \tag{9}$$

$$ \begin{aligned} S_{n+1}(t) &= \frac{1-\alpha}{B(\alpha)} \left\{ -\beta S_n(t) \frac{qE_n(t)+I_n(t)+A_n(t)}{N_n(t)} \right\} \\ &+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-y)^{\alpha-1} \left\{ -\beta S_n(y) \frac{qE_n(y)+I_n(y)+A_n(y)}{N_n(y)} \right\} dy \\ E_{n+1}(t) &= \frac{1-\alpha}{B(\alpha)} \left\{ \beta S_n(t) \frac{qE_n(t)+I_n(t)+A_n(t)}{N_n(t)} - \delta E_n(t) \right\} \\ &+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-y)^{\alpha-1} . \left\{ \beta S_n(y) \frac{qE_n(y)+I_n(y)+A_n(y)}{N_n(y)} - \delta E_n(y) \right\} dy \\ I_{n+1}(t) &= \frac{1-\alpha}{B(\alpha)} \{ p\delta E_n(t) - \gamma_1 I_n(t) \} \\ &+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-y)^{\alpha-1} \{ p\delta E_n(y) - \gamma_1 I_n(y) \} dy \\ A_{n+1}(t) &= \frac{1-\alpha}{B(\alpha)} \{ (1-p)\delta E_n(t) - \gamma_2 A_n(t) \} \\ &+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-y)^{\alpha-1} \{ (1-p)\delta E_n(y) - \gamma_2 A_n(y) \} dy \\ R_{n+1}(t) &= \frac{1-\alpha}{B(\alpha)} \{ \gamma_1 I_n(t) + \gamma_2 A_n(t) \} \\ &+ \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-y)^{\alpha-1} \{ \gamma_1 I_n(y) + \gamma_2 A_n(y) \} dy \\ C_{n+1}(t) &= \frac{1-\alpha}{B(\alpha)} \{ p\delta E_n(t) \} + \frac{\alpha}{B(\alpha)\Gamma(\alpha)} \int_0^t (t-y)^{\alpha-1} \{ p\delta E_n(y) \} dy \end{aligned} $$

Taking the limit for a large value of n, we expect to obtain the exact solution.

Using Picard–Lindelöf approach to check the existence

The proof is reached if one considers the following operator

$$ \begin{aligned} f_1(t,x) &= -\beta S(t) \frac{qE(t)+I(t)+A(t)}{N(t)} \\ f_2(t,x) &= \beta S(t) \frac{qE(t)+I(t)+A(t)}{N(t)} - \delta E(t) \\ f_3(t,x) &= p\delta E(t) - \gamma_1 I(t) \\ f_4(t,x) &= (1-p)\delta E(t) - \gamma_2 A(t) \\ f_5(t,x) &= \gamma_1 I(t) + \gamma_2 A(t) \\ f_6(t,x) &= p\delta E(t) \end{aligned} \tag{10}$$

It is clear that $f_1, f_2, f_3, f_4, f_5, f_6$ are contraction with respect to x for the first function, y for the second function, z for the third function and p, r, s are fourth, fifth, and sixth functions, respectively.

Let us consider

$$\begin{aligned} N_1 &= \sup_{C_{a,b_1}}\|f_1(t,x)\|, \quad N_2 = \sup_{C_{a,b_2}}\|f_2(t,y)\| \\ N_3 &= \sup_{C_{a,b_3}}\|f_3(t,z)\|, \quad N_4 = \sup_{C_{a,b_4}}\|f_4(t,p)\| \\ N_5 &= \sup_{C_{a,b_5}}\|f_5(t,r)\|, \quad N_6 = \sup_{C_{a,b_6}}\|f_6(t,s)\| \end{aligned} \tag{11}$$

where

$$\begin{aligned} C_{a,b_1} &= [t-a,t+a]\times[x-b_1,x+b_1] = A_1\times B_1 \\ C_{a,b_2} &= [t-a,t+a]\times[x-b_2,x+b_2] = A_1\times B_2 \\ C_{a,b_3} &= [t-a,t+a]\times[x-b_3,x+b_3] = A_1\times B_3 \\ C_{a,b_4} &= [t-a,t+a]\times[x-b_4,x+b_4] = A_1\times B_4 \\ C_{a,b_5} &= [t-a,t+a]\times[x-b_5,x+b_5] = A_1\times B_5 \\ C_{a,b_6} &= [t-a,t+a]\times[x-b_6,x+b_6] = A_1\times B_6 \end{aligned} \tag{12}$$

However, the fixed-point theorem of Banach space can be employed here together with the metric for our set of equations by inducing the uniform norm as

$$\|f(t)\|_\infty = \sup_{t\in t-a,\,t+a]}|f(t)| \tag{13}$$

The next operator is defined between the two functional spaces of continuous functions, and Picard's operator is defined as follows

$$O: C(A_1,B_1,B_2,B_3,B_4,B_5,B_6) \to C(A_1,B_1,B_2,B_3,B_4,B_5,B_6) \tag{14}$$

Defined as follows

$$\begin{aligned} OX(t) &= X_0(t) + X(t)\frac{1-\alpha}{B(\alpha)} \\ &\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-y)^{\alpha-1}F(y,X(y))dy \end{aligned} \tag{15}$$

where X is the given matrix

$$X(t) = \begin{cases} S(t) \\ E(t) \\ I(t) \\ A(t) \\ R(t) \\ C(t) \end{cases}, \quad X_0(t) = \begin{cases} g_1(t) \\ g_2(t) \\ g_3(t) \\ g_4(t) \\ g_5(t) \\ g_6(t) \end{cases}, \quad F(t,X(t)) = \begin{cases} f_1(t,x) \\ f_2(t,x) \\ f_3(t,x) \\ f_4(t,x) \\ f_5(t,x) \\ f_6(t,x) \end{cases} \tag{16}$$

Due to the fact that there is no disease that is able to kill the whole world population, also the fact that the number of targeted population is finite, we can assume that all the solutions are bounded within a period of time

$$\|x(t)\|_\infty \le \max\{b_1,b_2,b_3,b_4,b_5,b_6\} \tag{17}$$

$$\begin{aligned} &\|OX(t) - X_0(t)\| \\ &= \left\| F(t,X(t))\frac{1-\alpha}{B(\alpha)} + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-y)^{\alpha-1}F(y,X(y))dy \right\| \\ &\le \frac{1-\alpha}{B(\alpha)}\|F(t,X(t))\| + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-y)^{\alpha-1}\|F(y,X(y))\|dy \\ &\le \frac{1-\alpha}{B(\alpha)}N = \max\{N_1,N_2,N_3,N_4,N_5,N_6\} \\ &\quad + \frac{\alpha}{B(\alpha)}Na^\alpha < aN \le b = \max\{b_1,b_2,b_3,b_4,b_5,b_6\} \end{aligned} \tag{18}$$

Here, we request that

$$a < \frac{b}{N}$$

We next evaluate additionally the following

$$\|OX_1 - OX_2\|_\infty = \sup_{t\in A}|X_1 - X_2| \tag{19}$$

With the definition of the defined operator in hand, we produce the following

$$\|OX_1 - OX_2\| = \left\| \begin{array}{c} \{F(t,X_1(t)) - F(t,X_2(t))\}\frac{1-\alpha}{B(\alpha)} \\ + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-l)^{\alpha-1}\begin{Bmatrix} F(l,X_1(l)) \\ -F(l,X_2(l)) \end{Bmatrix}dl \end{array} \right\| \tag{20}$$

$$\begin{aligned} &\le \frac{1-\alpha}{B(\alpha)}\|F(t,X_1(t)) - F(t,X_2(t))\| \\ &\quad + \frac{\alpha}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-y)^{\alpha-1}\|F(l,X_1(y)) - F(l,X_2(y))\|dy \\ &\le \frac{1-\alpha}{B(\alpha)}q\|X_1(t) - X_2(t)\| \\ &\quad + \frac{\alpha q}{B(\alpha)\Gamma(\alpha)}\int_0^t (t-y)^{\alpha-1}\|X_1(y) - X_2(y)\|dy \\ &\le \left\{\frac{1-\alpha}{B(\alpha)}q + \frac{\alpha q a^\alpha}{B(\alpha)\Gamma(\alpha)}\right\}\|X_1(t) - X_2(t)\| \\ &\le aq\|X_1(t) - X_2(t)\| \end{aligned} \tag{21}$$

where $q < 1$. Since F is a contraction we have that $aq < 1$, the defined operator O is a contraction too. This shows that the system under investigation is a unique set of solution.

Obtention of specific solutions via iteration approach

Since the extended model is nonlinear, it is sometimes difficult to have it solved using analytical method; therefore, the need of an iterative approach is important. The method based on integral transform and iterative method will be used here to obtain a particular set of solutions for the extended model. The integral transform used here is the well-known Sumudu transform operator which has the properties of keeping the parity of the function. The following theorem is needed for further investigation, and the initial introduction of this theorem can be found in the study of Atangana and Koca.[12]

Theorem 2. Let $f \in H^1(a,b)$, $b>a$ and $\alpha \in [0,1]$, the Sumudu transform of ABC is given as[12]

$$ST\left\{{}_0^{ABC}D_t^{\alpha}(f(t))\right\} = \frac{B(\alpha)}{1-\alpha}\left(\alpha\Gamma(\alpha+1)E_\alpha\left(-\frac{1}{1-\alpha}p^\alpha\right)\right)\left(ST(f(t)) - f(0)\right) \tag{22}$$

Proof. Proof of the theorem can be found in the study of Atangana and Koca.[12]

To solve the above system (4), we apply the Sumudu transform of the Atangana–Baleanu fractional derivative of $f(t)$ on the system with both sides. Then, we obtain the below set

$$\begin{aligned}
&\frac{B(\alpha)}{1-\alpha}\left(\alpha\Gamma(\alpha+1)E_\alpha\left(-\frac{1}{1-\alpha}p^\alpha\right)\right)(ST(S(t)) - S(0)) = ST\left\{-\beta S(t)\frac{qE(t)+I(t)+A(t)}{N(t)}\right\}\\
&\frac{B(\alpha)}{1-\alpha}\left(\alpha\Gamma(\alpha+1)E_\alpha\left(-\frac{1}{1-\alpha}p^\alpha\right)\right)(ST(E(t)) - E(0)) = ST\left\{\beta S(t)\frac{qE(t)+I(t)+A(t)}{N(t)} - \delta E(t)\right\}\\
&\frac{B(\alpha)}{1-\alpha}\left(\alpha\Gamma(\alpha+1)E_\alpha\left(-\frac{1}{1-\alpha}p^\alpha\right)\right)(ST(I(t)) - I(0)) = ST\{p\delta E(t) - \gamma_1 I(t)\}\\
&\frac{B(\alpha)}{1-\alpha}\left(\alpha\Gamma(\alpha+1)E_\alpha\left(-\frac{1}{1-\alpha}p^\alpha\right)\right)(ST(A(t)) - A(0)) = ST\{(1-p)\delta E(t) - \gamma_2 A(t)\}\\
&\frac{B(\alpha)}{1-\alpha}\left(\alpha\Gamma(\alpha+1)E_\alpha\left(-\frac{1}{1-\alpha}p^\alpha\right)\right)(ST(R(t)) - R(0)) = ST\{\gamma_1 I(t) + \gamma_2 A(t)\}\\
&\frac{B(\alpha)}{1-\alpha}\left(\alpha\Gamma(\alpha+1)E_\alpha\left(-\frac{1}{1-\alpha}p^\alpha\right)\right)(ST(C(t)) - C(0)) = ST\{p\delta E(t)\}
\end{aligned} \tag{23}$$

Rearranging, we obtain following inequalities where $\lambda = -\frac{1}{1-\alpha}$

$$\begin{aligned}
ST(S(t)) &= S(0) + \frac{1-\alpha}{B(\alpha)(\alpha\Gamma(\alpha+1)E_\alpha(\lambda p^\alpha))}ST\left\{-\beta S(t)\frac{qE(t)+I(t)+A(t)}{N(t)}\right\}\\
ST(E(t)) &= E(0) + \frac{1-\alpha}{B(\alpha)(\alpha\Gamma(\alpha+1)E_\alpha(\lambda p^\alpha))}ST\left\{\beta S(t)\frac{qE(t)+I(t)+A(t)}{N(t)} - \delta E(t)\right\}\\
ST(I(t)) &= I(0) + \frac{1-\alpha}{B(\alpha)(\alpha\Gamma(\alpha+1)E_\alpha(\lambda p^\alpha))}ST\{p\delta E(t) - \gamma_1 I(t)\}\\
ST(A(t)) &= A(0) + \frac{1-\alpha}{B(\alpha)(\alpha\Gamma(\alpha+1)E_\alpha(\lambda p^\alpha))}ST\{(1-p)\delta E(t) - \gamma_2 A(t)\}\\
ST(R(t)) &= R(0) + \frac{1-\alpha}{B(\alpha)(\alpha\Gamma(\alpha+1)E_\alpha(\lambda p^\alpha))}ST\{\gamma_1 I(t) + \gamma_2 A(t)\}\\
ST(C(t)) &= C(0) + \frac{1-\alpha}{B(\alpha)(\alpha\Gamma(\alpha+1)E_\alpha(\lambda p^\alpha))}ST\{p\delta E(t)\}
\end{aligned} \tag{24}$$

We next obtain the following recursive formula

$$\begin{aligned}
S_{n+1}(t) &= S_n(0) + ST^{-1}\left\{\frac{1-\alpha}{B(\alpha)(\alpha\Gamma(\alpha+1)E_\alpha(\lambda p^\alpha))}ST\left\{-\beta S_n(t)\frac{qE_n(t) + I_n(t) + A_n(t)}{N_n(t)}\right\}\right\}\\
E_{n+1}(t) &= E_n(0) + ST^{-1}\left\{\frac{1-\alpha}{B(\alpha)(\alpha\Gamma(\alpha+1)E_\alpha(\lambda p^\alpha))}ST\left\{\beta S_n(t)\frac{qE_n(t) + I_n(t) + A_n(t)}{N_n(t)} - \delta E_n(t)\right\}\right\}\\
I_{n+1}(t) &= I_n(0) + ST^{-1}\left\{\frac{1-\alpha}{B(\alpha)(\alpha\Gamma(\alpha+1)E_\alpha(\lambda p^\alpha))}ST\{p\delta E_n(t) - \gamma_1 I_n(t)\}\right\}\\
A_{n+1}(t) &= A_n(0) + ST^{-1}\left\{\frac{1-\alpha}{B(\alpha)(\alpha\Gamma(\alpha+1)E_\alpha(\lambda p^\alpha))}ST\{(1-p)\delta E_n(t) - \gamma_2 A_n(t)\}\right\}\\
R_{n+1}(t) &= R_n(0) + ST^{-1}\left\{\frac{1-\alpha}{B(\alpha)(\alpha\Gamma(\alpha+1)E_\alpha(\lambda p^\alpha))}ST\{\gamma_1 I_n(t) + \gamma_2 A_n(t)\}\right\}\\
C_{n+1}(t) &= C_n(0) + ST^{-1}\left\{\frac{1-\alpha}{B(\alpha)(\alpha\Gamma(\alpha+1)E_\alpha(\lambda p^\alpha))}ST\{p\delta E_n(t)\}\right\}
\end{aligned} \tag{25}$$

And the solution of equation (25) is provided by

$$\begin{aligned}
S(t) &= \lim_{n\to\infty} S_n(t)\\
E(t) &= \lim_{n\to\infty} E_n(t)\\
I(t) &= \lim_{n\to\infty} I_n(t)\\
A(t) &= \lim_{n\to\infty} A_n(t)\\
R(t) &= \lim_{n\to\infty} R_n(t)\\
C(t) &= \lim_{n\to\infty} C_n(t)
\end{aligned} \tag{26}$$

Application of fixed-point theorem for stability analysis of iteration method

Let $(X, \|\cdot\|)$ be a Banach space and H a self-map of X. Let $y_{n+1} = g(H, y_n)$ be particular recursive procedure. Suppose that $F(H)$ is the fixed-point set of H and has at least one element and that y_n converges to a point $p \in F(H)$. Let $\{x_n\} \subseteq X$ and define $e_n = \|x_{n+1} - g(H, x_n)\|$. If $\lim_{n\to\infty} e^n = 0$ implies that $\lim_{n\to\infty} x^n = p$, then the iteration method $y_{n+1} = g(H, y_n)$ is said to be H-stable. Without any loss of generality, we must assume that our sequence $\{x_n\}$ has an upper boundary; otherwise, we cannot expect the possibility of convergence. If all these conditions are satisfied for $y_{n+1} = Hy_n$ which is known as Picard's iteration, consequently, the iteration will be H-stable. We shall then state the following theorem.

Theorem 3. Let $(X, \|\cdot\|)$ be a Banach space and H a self-map of X satisfying

$$\|H_x - H_y\| \le C\|x - H_x\| + c\|x - y\|$$

for all x, y in X where $0 \le C,\ 0 \le c < 1$. Suppose that H is Picard's H-stable.[21]

Now, we consider the recursive formula (25) with (4) below

$$\begin{aligned}
S_{n+1}(t) &= S_n(0) + ST^{-1}\left\{\Phi\cdot ST\left\{-\beta S_n(t)\frac{qE_n(t) + I_n(t) + A_n(t)}{N_n(t)}\right\}\right\}\\
E_{n+1}(t) &= E_n(0) + ST^{-1}\left\{\Phi\cdot ST\left\{\beta S_n(t)\frac{qE_n(t) + I_n(t) + A_n(t)}{N_n(t)} - \delta E_n(t)\right\}\right\}\\
I_{n+1}(t) &= I_n(0) + ST^{-1}\{\Phi\cdot ST\{p\delta E_n(t) - \gamma_1 I_n(t)\}\}\\
A_{n+1}(t) &= A_n(0) + ST^{-1}\{\Phi\cdot ST\{(1-p)\delta E_n(t) - \gamma_2 A_n(t)\}\}\\
R_{n+1}(t) &= R_n(0) + ST^{-1}\{\Phi\cdot ST\{\gamma_1 I_n(t) + \gamma_2 A_n(t)\}\}\\
C_{n+1}(t) &= C_n(0) + ST^{-1}\{\Phi\cdot ST\{p\delta E_n(t)\}\}
\end{aligned} \tag{27}$$

where $\Phi = \frac{1-\alpha}{B(\alpha)(\alpha\Gamma(\alpha+1)E_\alpha(\lambda p^\alpha))}$ is the fractional Lagrange multiplier.

Theorem 4. Let H be a self-map defined as

$$\begin{aligned}
H(S_n(t)) &= S_{n+1}(t) = S_n(t) + ST^{-1}\left\{\Phi\cdot ST\left\{-\beta S_n(t)\frac{qE_n(t) + I_n(t) + A_n(t)}{N_n(t)}\right\}\right\}\\
H(E_n(t)) &= E_{n+1}(t) = E_n(t) + ST^{-1}\left\{\Phi\cdot ST\left\{\beta S_n(t)\frac{qE_n(t) + I_n(t) + A_n(t)}{N_n(t)} - \delta E_n(t)\right\}\right\}\\
H(I_n(t)) &= I_{n+1}(t) = I_n(t) + ST^{-1}\{\Phi\cdot ST\{p\delta E_n(t) - \gamma_1 I_n(t)\}\}\\
H(A_n(t)) &= A_{n+1}(t) = A_n(t) + ST^{-1}\{\Phi\cdot ST\{(1-p)\delta E_n(t) - \gamma_2 A_n(t)\}\}\\
H(R_n(t)) &= R_{n+1}(t) = R_n(t) + ST^{-1}\{\Phi\cdot ST\{\gamma_1 I_n(t) + \gamma_2 A_n(t)\}\}\\
H(C_n(t)) &= C_{n+1}(t) = C_n(t) + ST^{-1}\{\Phi\cdot ST\{p\delta E_n(t)\}\}
\end{aligned} \tag{28}$$

is H-stable in $L^1(a, b)$ if

$$\begin{aligned}&\|H(S_n(t)) - H(S_m(t))\| \le \|S_n(t) - S_m(t)\|(1 - \beta C(\Theta))\\&\|H(E_n(t)) - H(E_m(t))\| \le \|E_n(t) - E_m(t)\|(1 + \beta d(\Theta) - \delta e(\Theta))\\&\|H(I_n(t)) - H(I_m(t))\| \le \|I_n(t) - I_m(t)\|(1 + f(\Theta)p\delta - g(\Theta)\gamma_1)\\&\|H(A_n(t)) - H(A_m(t))\| \le \|A_n(t) - A_m(t)\|(1 + (1-p)\delta h(\Theta) - \gamma_2 r(\Theta))\\&\|H(R_n(t)) - H(R_m(t))\| \le \|R_n(t) - R_m(t)\|(1 + \gamma_1 t(\Theta) - \gamma_2 k(\Theta))\\&\|H(C_n(t)) - H(C_m(t))\| \le \|C_n(t) - C_m(t)\|(1 + p\delta r(\Theta))\end{aligned} \tag{29}$$

Proof. The first step of the proof shows that H has a fixed point. To achieve this, we evaluate the following for all $(n, m) \in \mathbb{N} \times \mathbb{N}$

$$\begin{aligned}H(S_n(t)) - H(S_m(t)) &= S_n(t) - S_m(t)\\&+ ST^{-1}\left\{\Phi \cdot ST\left\{-\beta S_n(t)\frac{qE_n(t) + I_n(t) + A_n(t)}{N_n(t)}\right\}\right\}\\&- ST^{-1}\left\{\Phi \cdot ST\left\{-\beta S_m(t)\frac{qE_m(t) + I_m(t) + A_m(t)}{N_m(t)}\right\}\right\}\end{aligned} \tag{30}$$

Let us consider equality (30) and apply norm on both sides and without loss of generality

$$\|H(S_n(t)) - H(S_m(t))\| = \left\| \begin{aligned}&S_n(t) - S_m(t)\\&+ ST^{-1}\left\{\Phi \cdot ST\left\{\begin{aligned}&-\beta S_n(t)\frac{qE_n(t) + I_n(t) + A_n(t)}{N_n(t)}\\&-\left(-\beta S_m(t)\frac{qE_m(t) + I_m(t) + A_m(t)}{N_m(t)}\right)\end{aligned}\right\}\right\}\end{aligned}\right\| \tag{31}$$

$$\le \|S_n(t) - S_m(t)\| + \left\|ST^{-1}\{\Phi \cdot ST\{-\beta S_n(t)K_n(t) + \beta S_m(t)K_m(t)\}\}\right\| \tag{32}$$

where

$$\begin{aligned}K_n(t) &= \frac{qE_n(t) + I_n(t) + A_n(t)}{N_n(t)}\\K_m(t) &= \frac{qE_m(t) + I_m(t) + A_m(t)}{N_m(t)}\end{aligned} \tag{33}$$

Because $N_n(t)$ and $N_m(t)$ are total population size, we can consider equality as below

$$\|H(S_n(t)) - H(S_m(t))\| \le \|S_n(t) - S_m(t)\|(1 - \beta C(\Theta)) \tag{34}$$

where $C(\Theta)$ is the $ST^{-1}\{\Phi \cdot ST\}$. With same idea, we have following

$$\begin{aligned}&\|H(E_n(t)) - H(E_m(t))\| \le \|E_n(t) - E_m(t)\|(1 + \beta d(\Theta) - \delta e(\Theta))\\&\|H(I_n(t)) - H(I_m(t))\| \le \|I_n(t) - I_m(t)\|(1 + f(\Theta)p\delta - g(\Theta)\gamma_1)\\&\|H(A_n(t)) - H(A_m(t))\| \le \|A_n(t) - A_m(t)\|(1 + (1-p)\delta h(\Theta) - \gamma_2 r(\Theta))\\&\|H(R_n(t)) - H(R_m(t))\| \le \|R_n(t) - R_m(t)\|(1 + \gamma_1 t(\Theta) - \gamma_2 k(\Theta))\\&\|H(C_n(t)) - H(C_m(t))\| \le \|C_n(t) - C_m(t)\|(1 + p\delta r(\Theta))\end{aligned} \tag{35}$$

This completes the proof.

Conclusion

Many epidemiological models aim to describe complicated physical problems. To explain the spread of a given sickness, modellers use the concept of differentiation to predict the future behaviours of the spread. However, in the last passed years, many researchers rely on the concept of rate of change that is based on the Newton law. Other researchers make use of the concept of power law that is based on the concept of fractional differentiation. The fractional differentiation was introduced to model some complicated physical aspect; however, they have been found not quite efficient when modelling the spread of some diseases. Recently, due to the application of the Mittag-Leffler function in many fields of science and engineering, the fractional differentiation based on the generalized Mittag-Leffler function was constructed, and some applications were made with great success. In this work, we have extended the model of H1N1 to the concept of fractional differentiation based on the Mittag-Leffler function. We studied the existence of the generalized model using the fixed-point theorem. We presented the derivation of the solution using the Sumudu transform, and the stability analysis of the method is validated via the t-stable approach.

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References

1. Baskonus HM and Bulut H. On the numerical solutions of some fractional ordinary differential equations by fractional Adams-Bashforth-Moulton method. *Open Math* 2015; 13: 547–556.

2. Bulut H, Baskonus HM and Belgacem FBM. The analytical solutions of some fractional ordinary differential equations by Sumudu transform method. *Abstr Appl Anal* 2013; 2013: 203875-1–203875-6.
3. Baskonus HM, Mekkaoui T, Hammouch H, et al. Active control of a Chaotic fractional order economic system. *Entropy* 2015; 17: 5771–5783.
4. Bhrawy AH, Ezz-Eldien SS, Doha EH, et al. Solving fractional optimal control problems within a Chebyshev-Legendre operational technique. *Int J Control*. Epub ahead of print 1 February 2017. DOI: 10.1080/00207179.2016.1278267.
5. Kumar D, Singh J and Baleanu D. A hybrid computational approach for Klein–Gordon equations on Cantor sets. *Nonlinear Dynam* 2017; 87: 511–517.
6. El-Sayed AMA, Elsaid A, El-Kalla IL, et al. A homotopy perturbation technique for solving partial differential equations of fractional order in finite domains. *Appl Math Comput* 2012; 218: 8329–8340.
7. Golmankhaneh AK, Golmankhaneh AK and Baleanu D. On nonlinear fractional Klein–Gordon equation. *Signal Process* 2011; 91: 446–451.
8. Sambandham B and Vatsala A. Basic results for sequential Caputo fractional differential equations. *Mathematics* 2015; 3: 76–91.
9. Arafa AAM, Rida SZ and Mohamed H. Homotopy analysis method for solving biological population model. *Commun Theor Phys* 2011; 56: 797–800.
10. Gorenflo R, Mainardi F, Scalas E, et al. Fractional calculus and continuous time finance III: the diffusion limit. In: Gorenflo R, Mainardi F, Scalas E, et al. (eds) *Mathematical finance, trends in mathematics*. Basel: Birkhauser, 2001, pp.171–180.
11. Atangana A and Baleanu D. New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model. *Therm Sci*. Epub ahead of print January 2016. DOI: 10.2298/TSCI160111018A.
12. Atangana A and Koca I. Chaos in a simple nonlinear system with Atangana-Baleanu derivatives with fractional order. *Chaos Soliton Fract* 2016; 89: 447–454.
13. Atangana A and Koca I. On the new fractional derivative and application to nonlinear Baggs and Freedman model. *J Nonlinear Sci Appl* 2016; 9: 2467–2480.
14. Schneider WR and Wyss W. Fractional diffusion and wave equations. *J Math Phys* 1989; 30: 134–144.
15. Hristov J. Transient heat diffusion with a non-singular fading memory: from the Cattaneo constitutive equation with Jeffrey's kernel to the Caputo-Fabrizio time-fractional derivative. *Therm Sci* 2016; 20: 765–770.
16. Hristov J. Steady-state heat conduction in a medium with spatial non-singular fading memory: derivation of Caputo-Fabrizio space-fractional derivative with Jeffrey's kernel and analytical solutions. *Therm Sci*. Epub ahead of print January 2016. DOI: 10.2298/TSCI160229115H.
17. Yang X-J, Zhang ZZ and Srivastava HM. Some new applications for heat and fluid flows via fractional derivatives without singular kernel. *Therm Sci* 2016; 20: 833–839.
18. Yang X-J, Srivastava HM and Machado JAT. A new fractional derivative without singular kernel: application to the modelling of the steady heat flow. *Therm Sci* 2016; 20: 753–756.
19. Gao F and Yang X-J. Fractional Maxwell fluid with fractional derivative without singular kernel. *Therm Sci* 2016; 20: 871–877.
20. Tan X, Yuan L, Zhou J, et al. Modeling the initial transmission dynamics of influenza A H1N1 in Guangdong Province, China. *Int J Infect Dis* 2013; 17: 479–484.
21. Odibat ZM and Momani S. Application of variational iteration method to nonlinear differential equation of fractional order. *Int J Nonlinear Sci* 2006; 7: 27–34.